Homework 6: Solutions to exercises not appearing in Pressley

Math 120A

- (4.2.14) Let $f(x) = x^3 + 3(y^2 + z^2)^2 - 2$. Then the gradient of f is $\nabla f = (3x^2, 12y(y^2 + z^2), 12z(y^2 + z^2))$, which is zero only when $x = y = z = 0$. Since $(0, 0, 0)$ is not a solution to $f(x) = 0$, we see that $\nabla f \neq 0$ at all solutions to $f(x) = 0$, so the level set $\{(x, y, z) : f(x) = 0\}$ is a surface.

- (4.4.5) We have a parametrization of the torus

  $$\sigma(\theta, \phi) = ((a + b\cos\theta)\cos\phi, (a + b\cos\theta)\sin\phi, b\sin\theta)$$

  whose restriction to $(0, 2\pi) \times (0, 2\pi)$ is a surface patch containing $\theta = \phi = \frac{\pi}{4}$. The partial derivatives of σ are

  $$\begin{aligned}\sigma_\theta &= (-b\sin\theta\cos\phi, -b\sin\theta\sin\phi, b\cos\theta)\\ \sigma_\phi &= (-(a + b\cos\theta)\sin\phi, (a + b\cos\theta)\cos\phi, 0)\end{aligned}$$

  At $\theta = \phi = \frac{\pi}{4}$, these vectors are

  $$\begin{aligned}\sigma_u &= \left(\frac{-b}{2}, \frac{-b}{2}, \frac{b}{\sqrt{2}}\right) = \frac{b}{2}(-1, -1, \sqrt{2})\\ \sigma_v &= \left(\frac{-a}{\sqrt{2}} - \frac{b}{2}, \frac{a}{\sqrt{2}} + \frac{b}{2}, 0\right) = \left(\frac{a}{\sqrt{2}} + \frac{b}{2}\right)(-1, 1, 0)\end{aligned}$$

  Therefore the normal vector to the plane spanned by σ_θ and σ_ϕ is

  $$(-1, -1, \sqrt{2}) \times (-1, 1, 0) = (-\sqrt{2}, -\sqrt{2}, -2) = -\sqrt{2}(1, 1, \sqrt{2})$$

  Ergo the tangent plane is $x + y + \sqrt{2}z = 0$.

- (Question 3) A surface patch for $\mathbb{R}^2$ is the identity map $\sigma_1 : \mathbb{R}^2 \to \mathbb{R}^2 \subseteq \mathbb{R}^3$ with $(x, y) \mapsto (x, y, 0)$. Then if σ is the parametrization of the torus from (4.2.5), we see that $\sigma^{-1} \circ f \circ \sigma_1(\phi, \theta) = (2\pi x, 2\pi y)$ wherever it is defined. The Jacobian of this map is $2\pi I$ everywhere, where I is the 2×2 identity matrix, and in particular always invertible. Therefore $D_{\mathbf{p}} f$ is an invertible linear matrix everywhere, and we conclude that f is a local diffeomorphism.